ON SOME TECHNIQUES FOR SOLVING EXTREME PROBLEMS

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ABSTRACT

This article will focus on an algebraic, analytical and geometric method that can be used to solve many extreme problems. Of course, there are other ways to solve all these problems. But in each case, come up with your own way. Here, they are all obtained by a comer method.

KEYWORDS: *Largest Value, Smallest Value, Linear Function, Circle, Segment, Inequality, Expression, Collinear.*

INTRODUCTION

1. About discriminant recall that inequality

$$ax^2 + bx + c \geq 0 \qquad (a > 0)$$

Holds for all *x* if and only if

$$b^2 - 4ac < 0$$

Inequality

$$ax^2 + bx + c < 0$$

Has solutions for $a > 0$ is and only is the corresponding equation $ax^2 + bx + c = 0$ has 2 different roots, i.e., for $b^2 - 4ac > 0$.

The solution to the following well – known problem is quite well illustrated by a solution method based on the listed properties of quadratic inequalities.

Task 1. Find the largest of the values of *Z* for which there are numbers *x, y* satisfying the equation

$$2x^2 + 2y^2 + z^2 + xy + xz + yz = 4$$

Solution. The discriminant of a quadratic equation with respect to *x* with coefficients depending on *y* and *z* mast be non-negative, i.e. the inequality must hold

$$D=(y+z)^2-16y^2-8yz-8z^2+32\geq 0$$

or

$$15y^2+6yz+7z^2-32\leq 0$$

a quadratic inequality with respect to *y* has a solution only if

$$30z^2-60(7z^2-32)\geq 0$$
$$3z^2-5(7z^2-32)\geq 0$$
$$3z^2-35z^2+160\geq 0$$
$$-32z^2+160\geq 0$$
$$z^2\leq 5$$

so, $-\sqrt{5}\leq z\leq\sqrt{5}$, so *z* can't be more than $\sqrt{5}$. If $z=\sqrt{5}$, then the inequality has a unique solution for *y*. but then there exists an *x* (also unique) such that the triple

(*x, y, z*) satisfies the equation

$$2x^2+2y^2+z^2+xy+xz+yz=4$$

The next task, although outwardly, different from the previous one, can be easily reduced to it.

Task 2. The numbers *x, y, z* are such that $x^2+2y^2+z^2=2$. What is the largest value that the expression can be $2x+y-z$?

Solution. Let $t=2x+y-z$. Then, after substituting $z=2x+y-t$ into the equation, we arrive at a problem that differs from the previous one only in the values of the coefficients, i.e.

$$x^2+2y^2+(2x+y-t)^2=2$$

$$x^2+2y^2+4x^2+y^2+t^2+4xy-4xt-2yt=2,$$

$$5x^2+4(y-t)x+3y^2+t^2-2yt-2=0$$

A quadratic equation with respect to *x* has *a* solution only if

$$D=16(y-t)^2-20(3y^2+t^2-2yt-2)\geq 0,$$

$$4(y^2-2yt+t^2)-5(3y^2+t^2-2yt-2)\geq 0,$$

$$4y^2-8yt+4t^2-15y^2-5t^2+10yt+10\geq 0,$$

$$-11y^2+2yt-t^2+10\geq 0$$

A quadratic inequality with respect to *y* has a solution only if

$$4t^2 + 44(-t^2 + 10) \geq 0,$$
$$t^2 - 11t^2 + 110 \geq 0,$$
$$-10t^2 \geq -110$$
$$t^2 \leq 11$$
$$|t| \leq \sqrt{11}$$
$$t_{\max} = \sqrt{11}$$

Task 3. Find the smallest value accepted by $x + 5y$, if $x > 0$, $y > 0$ and $x^2 - 6xy + y^2 + 21 \leq 0$

Solution. Let $t = x + 5y$, then $x = t - 5y$. After substitution and transformations, we arrive at the problem of finding the smallest positive. Value of *"t"* for which the corresponding inequality quadratic with respect to *"y"* will have a solution.

$$(t - 5y)^2 - 6(t - 5y)y + y^2 + 21 \leq 0,$$
$$t^2 - 10ty + 25y^2 - 6ty + 30y^2 + y^2 + 21 \leq 0,$$
$$56y^2 - 16ty + t^2 + 21 \leq 0,$$
$$D = 256t^2 - 224(t^2 + 21) \geq 0,$$
$$8t^2 - 7(t^2 + 21) \geq 0$$
$$t^2 \geq 147$$
$$|t| \geq 7\sqrt{3}$$
$$t_{\min} = 7\sqrt{3}$$

Task 4. For any positive numbers a, b, c, d find the smallest value expression

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b}$$

Lemma. Cauchy – Bunyakovsky inequality: for any numbers $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$

$$(x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2) \geq (x_1 y_1 + \dots + x_n y_n)^2$$

To prove the validity of the lemma, it suffices to note that the quadratic function

$$P(t) = (x_1 + y_1 t)^2 + \dots + (x_n + y_n t)^2$$

is nonnegative for all *t* and write down the nonpositiveness of its discriminant that follows from this, i.e.,

$$(y_1^2 + \dots + y_n^2)t^2 + 2(x_1 y_1 + \dots + x_n y_n)t + x_1^2 + \dots + x_n^2 \geq 0$$
$$4(x_1 y_1 + \dots + x_n y_n)^2 - 4(x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2) \leq 0$$

or

$$\left(x_1^2+\ldots+x_n^2\right)\left(y_1^2+\ldots+y_n^2\right)\geq\left(x_1y_1+\ldots+x_ny_n\right)^2$$

Now we can apply the lemma.

Solution. Put $x_1=\sqrt{a/b+c}$,

$x_2=\sqrt{b/c+d}$, $x_3=\sqrt{c/d+a}$, $x_4=\sqrt{d/a+b}$ and

$y_1=\sqrt{a/b+c}$, $y_2=\sqrt{b/c+d}$, $y_3=\sqrt{c/d+a}$, $y_4=\sqrt{d/a+b}$

By virtue of the Cauncy-Bunyakovsky inequality, which was proved above

$$\left(x_1^2+x_2^2+x_3^2+x_4^2\right)\left(y_1^2+y_2^2+y_3^2+y_4^2\right)=\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)$$

$$\left(2ac+2bd+ab+bc+cd+da\right)\geq\left(x_1y_1+x_2y_2+x_3y_3+x_4y_4\right)^2=$$

$$=\left(a+b+c+d\right)^2 \qquad (*)$$

Obvious, $a^2+c^2\geq 2ac,\ b^2+d^2\geq 2bd,$ so

$$\left(a+b+c+d\right)^2\geq 4ac+4bd+2\left(ab+bc+cd+da\right)$$

Substituting this estimate into (*), we obtain

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2 \qquad (*)$$

Mince the smallest value of the required expression is 2.

(**) is a special case of the inequality

$$\frac{a^1}{a_2+a_3}+\ldots+\frac{a_{n-1}}{a_n+a_1}+\frac{a_n}{a_1+a_2}\geq\frac{n}{2}\left(at\ \ a_i>0\right)$$

Which in 1954 the American mathematician Shapiro proposed to prove.

In our problem $n=4$.

2. Conditional extremum solved by a non-classical method

Task 5. Find the smallest value of expression

$$\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+x}$$

if $x,y,z>0$ and $\sqrt{xy}+\sqrt{yz}+\sqrt{zx}=1$

Solution. Note first that

$$\frac{x^2}{x+y}=x-\frac{xy}{x+y}\geq x-\frac{xy}{2\sqrt{xy}}=x-\frac{\sqrt{xy}}{2}$$

$$\frac{y^2}{y+z} \ge y - \frac{\sqrt{yz}}{2}, \quad \frac{z^2}{z+x} \ge z - \frac{\sqrt{zx}}{2}$$

and denote the given expression by *A*, we get that $A \ge x + y + z - \frac{1}{2}.$

By virtue of the well – known inequality

$$a^2 + b^2 + c^2 \ge ab + bc + ac$$

we have

$$A \ge \sqrt{xy} + \sqrt{yz} + \sqrt{zx} - \frac{1}{2}, \text{ i.e. } A \ge \frac{1}{2}$$

For $x = y = z = \frac{1}{3}$ we get $A = \frac{1}{2}$, so the smallest value of *A* is $\frac{1}{2}$.

Task 6. Find the smallest value of expression

$$\frac{1}{x^3 + y^3 + xyz} + \frac{1}{y^3 + z^3 + xyz} + \frac{1}{z^3 + x^3 + xyz}$$

if $x, y, z > 0$ and $xyz = 1$

Solution. If is easy to verity that for $x > 0, y > 0, z > 0$ the inequality $x^2 + y^3 \ge xy(x + y)$, so that the first term in the expression is less than or equal to $\frac{1}{xy(x+y+z)}$, so this expression does not exceed the sum

$$\frac{1}{x+y+z} \cdot \left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right) = \frac{1}{xyz} = 1$$

Task 7. Find smallest function value

$$y = \sqrt{2x^2 - 2x + 1} + \sqrt{2x^2 - (\sqrt{3} - 1)x + 1} + \sqrt{2x^2 - (\sqrt{3} + 1)x + 1}$$

First solution. Since

$$y = \sqrt{x^2 + (x-1)^2} + \sqrt{\left(x - \frac{\sqrt{3}}{2}\right)^2 + \left(x + \frac{1}{2}\right)^2} + \sqrt{\left(x - \frac{\sqrt{3}}{2}\right)^2 + \left(x - \frac{1}{2}\right)^2}$$

then for points

$$D(x, x), \; A(0,1), \; B\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \; C\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right),$$

$y = DA + DB + DC$, and, as is easy to verify, *ABC* is a regular triangle with center at the point $O(0,0)$.

As you know, the center has the smallest sum of distances to the vertices of a regular triangle, that is, the sum of $DA+DB+DC$ takes the smallest value for point $D(0,0)$, and therefore $\min y = y(0) = 3$.

Second solution. For vectors

$$\bar{a} = (1-2x, 1),\ \bar{b} = \left(x+1,\ 1-x\sqrt{3}\right),$$
$$\bar{c} = \left(x+1,\ 1+x\sqrt{3}\right)$$

we have $\bar{a}+\bar{b}+\bar{c} = (3;3)$

$$\left|\bar{a}+\bar{b}+\bar{c}\right| = 3\sqrt{2}$$

$$\left|\bar{a}\right| = \sqrt{(1-2x)^2+1},\quad \left|\bar{b}\right| = \sqrt{(x+1)^2+\left(1-x\sqrt{3}\right)^2}$$

$$\left|\bar{c}\right| = \sqrt{(x+1)^2+\left(1+x\sqrt{3}\right)^2}$$

$$\left|\bar{a}\right|+\left|\bar{b}\right|+\left|\bar{c}\right| = \sqrt{2\left[x^2+(x-1)^2\right.} + \sqrt{2\left[\left(x-\frac{\sqrt{3}}{2}\right)^2+\left(x+\frac{1}{2}\right)^2\right.} +$$

$$\sqrt{2\left[\left(x-\frac{\sqrt{3}}{2}\right)^2+\left(x-\frac{1}{2}\right)^2\right.} = \sqrt{2}y$$, it is know that $\left|\bar{a}\right|+\left|\bar{b}\right|+\left|\bar{c}\right| \ge \left|\bar{a}+\bar{b}+\bar{c}\right|$

$y\sqrt{2} \ge 3\sqrt{2}$ and therefore $y \ge 3$.

Task 8. Find the most value expression

$$\frac{a}{bc+1}+\frac{b}{ac+1}+\frac{c}{ab+1}$$

for any positive numbers *a, b, c* not exceeding 1.

Solution. Note at once that as *a, b, c* are condition completely symmetrical, we may assume that the

$$0 \le a \le b \le c \le 1$$

Since the $(1-a)(1-b) \ge 0$, then $a+b \le 14ab \le 1+2ab$.

Therefore $a+b+c \le a+b+1 \le 2+2ab$ since $1+ab \le 1+ac \le 1+bc$.

$$\frac{a}{bc+1}+\frac{b}{ac+1}+\frac{c}{ab+1} \le \frac{a+b+c}{1+ab} < 2$$

3. The sum of minima and at least amount of

The arbitrary function $f(x)$ and $g(x)$ defined, say, on a segment $[a,b]$, reach their lows in different points. Equality $\min f(x)+\min g(x)\le \min(f(x)+g(x))$. In fact, a minimum of the total functions $f(x)+g(x)$ is achieved in some point segment $[a,b]$. In this point value function $f(x)$ and $g(x)$ not less than their lows. It is clear that the inequality turns into a equality, if minima $f(x)$ and $g(x)$ achieved in the same point. Exactly the same situation is the case of *n* function:

$$\min f_1(x)+\ldots+\min f_n(x)\le \min(f_1(x)+\ldots+f_n(x))$$

The basic principle is that the sum of the minima of several functions does not exceed the minimum of their sum, and it remains valid for the case of functions of several variables. Moreover, his proof in this case does not change.

Consider the simplest linear function of two variables:

$$f(x;y)=ax+by$$

Surprisingly, but using this elementary function, you can get quite complicated inequalities

Task 9. Prove the inequalities

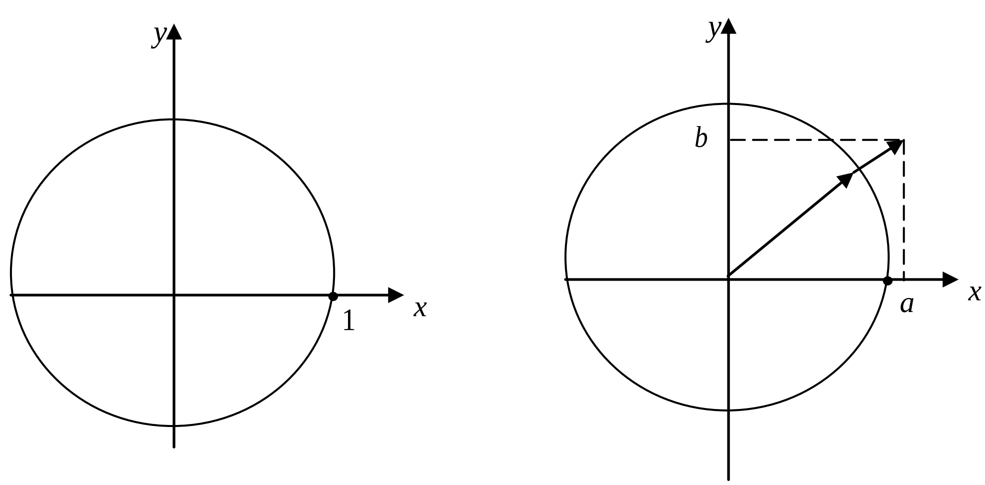
$$\sqrt{a_1^2+b_1^2}+\ldots+\sqrt{a_n^2+b_n^2}\ge\sqrt{(a_1+\ldots+a_2)^2+(b_1+\cdots+b_n)^2}$$

Solution. Consider *n* functions

$$f_1(x;y)=a_1x+b_1y,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_n(x;y)=a_nx+b_ny$$

The most important thing is to choose the right sets on which we have are considering our functions. After all, our functions are defend on a plane and we can find minimums on any figure.

Consider a circle of radius 1 centered at the origin (fig. 1).



$x^2 + y^2 = 1$

fig. 1. fig. 2.

What does the geometrically linear function $f(x, y) = ax + by$ mean? This is the dot product of vectors with coordinates $(a;b)$ and (x, y). But the dot product is the product of the lengths of the vectors and the cosine of the angle between them. The cosine of the angle is the minimum for collinear vectors, and the length of the vector $(x; y)$ does not change if the end of the vector lies on a circle. Hence, the minimum of the function $f(x; y)$ is achieved on a vector with coordinates $(x_0; y_0)$ collinear $(\overline{a;b})$ (fig. 2).

Such a vector is

$(\overline{x_0;\ y_0}) = \left(\frac{a}{\sqrt{a^2+b^2}};\ \frac{b}{\sqrt{a^2+b^2}}\right)$, and the dot product

$$(\overline{x_0;\ y_0}) = (\overline{a;b}) = \sqrt{a^2+b^2}$$

so, in the case of a circle

$$\min(ax+by) = \sqrt{a^2+b^2}$$

Not it costs us nothing to prove the inequality of problem 9. On the lest, it

contains the minimum values of the function $f_1, \ldots, f_n$ on the circle, and on the right, the minimum of their sum.

Task 10. Find the largest value of the function $z = 5x + \sqrt{11}y$ on the set of solutions of the system

$$x^2 + y^2 = 337^2$$
$$x \ge 0$$
$$y \ge 0$$

Solution. $(5x+\sqrt{11}y)^2 + (\sqrt{11}x - 5y)^2 = 25x^2 - 10\sqrt{11}xy + 11y^2 + 11x^2 - 10\sqrt{11}xy +$

$+25y^2 = 36(x^2+y^2) = 36 \cdot 337^2 = (6 \cdot 337)^2 = 2022^2$

$(5x+\sqrt{11}y)^2 = 2022^2 - (\sqrt{11}x - 5y)^2 \le 2022^2$

$5x + \sqrt{11}y \le 2022$

$\max(5x+\sqrt{11}y) = 2022$

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