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SOLVE MATHEMATICAL OLYMPEAD PROBLEMS USING THE STOLZ'S THEOREM

Ismatullayeva Gulrukh*

*3rd year student of the Faculty of Mathematics,
Samarkand State University,
Samarkand, UZBEKISTAN
Email id: GulruhIsmatullayeva0@gmail.com

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ABSTRACT

This thesis presents and solves the problems of the Mathematical Olympiad that can be solved using the Stolz-Cesàro theorem. The theorem is named after mathematicians. Otto Stolz and Ernesto Cesaro, who stated and proved it for the first time. This theorem can also be used not only to solve the problems of the Mathematical Olympiad among university students, but also to solve the problems of the Olympiad for school and high school students.

KEYWORDS: *Sequences, Limit, Stolz's Theorem, Hopital's Rule*

INTRODUCTION

In mathematics the Stolz-Cesaro theorem is a criterion for proving the convergence of a sequence. The theorem is named after mathematicians. Otto Stolz and Ernesto Cesaro, who stated and proved it for the first time. The Stolz-Cesaro theorem can be viewed as a generalization of the Cesaro mean but also as a Hopital's rule for sequences.

This theorem can also be used not only to solve the problems of the Mathematical Olympiad among university students, but also to solve the problems of the Olympiad for school and high school students. The following is a summary of this theorem and some of the Mathematical Olympiad problems that can be solved on the basis of this theorem

Stolz-Cesaro theorem

The famous Stolz-Cesaro theorem states that if y_n is a strictly increasing sequence

$y_{n+1} > y_n \qquad n=1,2,3, \ldots$

with

$\lim_{n\to\infty} y_n = +\infty$

and

$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L \in R,$

then we have

$$\lim_{n\to\infty} \frac{x_n}{y_n} = L$$

Problem 1

Prove that the limit is 1.

$$\lim_{n\to\infty}\frac{1}{\ln(n+1)}*\sum_{i=1}^{n} i^{-1}=1$$

Solution

We use the Stolz-Cesaro theorem to calculate the limit

$x_n=\sum_{i=1}^{n} i^{-1}$

$y_n=\ln(n+1)$

(i) $y_{n+1}>y_n$ n = 1,2,3,…..

$y_{n+1}=\ln(n+2)>y_n=\ln(n+1)$, n=1,2,3, ..

(ii) $\lim\limits_{n\to\infty} y_n=\lim\limits_{n\to\infty}\ln(n+1)=+\infty$

L =

$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=\lim_{n\to\infty}\frac{\sum_{i=1}^{n+1} i^{-1}-\sum_{i=1}^{n} i^{-1}}{\ln(n+2)-\ln(n+1)}=\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\ln(1+\frac{1}{n+1})}=\lim_{n\to\infty}\frac{1}{(n+1)\ln(1+\frac{1}{n+1})}=$
$\lim_{n\to\infty}\frac{1}{\ln(1+\frac{1}{n+1})^{n+1}}=\frac{1}{lne}=1$

Based on the conclusion of the theorem

L = $\lim_{n\to\infty}\frac{1}{\ln(n+1)}*\sum_{i=1}^{n} i^{-1}=1$

Problem 2

If a>0, evaluase $\lim_{n\to\infty}\frac{\sqrt{a}+\sqrt[2]{a}+\sqrt[3]{a}\ldots..+\sqrt[n]{a}-n}{\ln n}$

Solution

$a_n=\sqrt{a}+\sqrt[2]{a}+\sqrt[3]{a}\ldots\ldots+\sqrt[n]{a}-n$

b_n = lnn

(i) $b_{n+1}>b_n$ n=1,2,3, ….

ln(n+1)>lnn

(ii) $\lim\limits_{n\to\infty} b_n=\lim\limits_{n\to\infty} lnn=+\infty$

L = $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim_{n\to\infty}\frac{\sqrt[n+1]{a}-(n+1)+n}{\ln(n+1)-\ln(n)}=\lim_{n\to\infty}\frac{a^{\frac{1}{n+1}}-1}{\ln(n+1)}*\frac{\frac{1}{n}}{\ln\left(1+\frac{1}{n}\right)}*\frac{n}{n+1}=lna$

L = $\lim_{n\to\infty}\frac{\sqrt{a}+\sqrt[2]{a}+\sqrt[3]{a}\ldots..+\sqrt[n]{a}-n}{\ln n}=lna$

Problem3

$$\lim_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{(n)!} - ?$$

Determine the value of Lesenjeri limit:

Solution

$a_n = \sqrt[n]{(n)!}$

$b_n = n$

(i) $b_{n+1} > b_n$ n = 1,2,3, ….

$n+1 > n$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} n = +\infty$

$$L = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{(n)!}}{n+1-n}$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\sqrt[n]{(n)!}}{n}$$

According to the Sterling formula

$$\lim_{n\to\infty} \frac{n!}{(\frac{n}{e})^n\sqrt{2\pi n}} = 1 \Rightarrow \quad n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(n)!}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{(\frac{n}{e})^n\sqrt{2\pi n}}}{n} = \lim_{n\to\infty} \frac{\frac{n}{e}\sqrt[2n]{\sqrt{2\pi n}}}{n} = \lim_{n\to\infty} \frac{\sqrt[2n]{\sqrt{2\pi n}}}{e} = \frac{1}{e}$$

$$L = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{(n)!}}{n+1-n} = \frac{\sqrt[n]{(n)!}}{n} = \frac{1}{e}$$

$$\lim_{n\to\infty} (\sqrt[n+1]{(n+1)!} - \sqrt[n]{(n)!}) = e^{-1}$$

Problem4

$\forall\, n \epsilon N, \quad x_n > 0$

$$\lim_{n\to\infty} (\sqrt[n]{(x_n)} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$

Solution

$$\sqrt[n]{(x_n)} = \sqrt{x_q * \frac{x_2}{x_1} * \frac{x_3}{x_2} * \dots * \frac{x_n}{x_{n-1}}};$$

$c_1 = x_1$ $\quad c_2 = \frac{x_2}{x_1}$ $\quad c_3 = \frac{x_3}{x_2}$ $\quad \dots.$ $\quad c_n = \frac{x_n}{x_{n-1}}$

$$\lim_{n\to\infty} (\sqrt[n]{(x_n)} = \sqrt[n]{c_1 * c_2 * \dots .* c_n} = \lim_{n\to\infty} e^{\lim_{n\to\infty} ln \sqrt[n]{c_1 * c_2 * \dots .* c_n}},$$

Here for e× function is continuous, it can be solved according to limit degree in (x∈R)

$$\underset{n\to\infty}{lim} e^{ln\sqrt[n]{c_1 * c_2 * \ldots * c_n}} = e^{\lim_{n\to\infty} ln\sqrt[n]{c_1 * c_2 * \ldots * c_n}} = \lim_{n\to\infty} e^{\lim_{n\to\infty}\frac{1}{n}\ln(c_1 * c_2 * \ldots * c_n)} = e^{\lim_{n\to\infty}\frac{1}{n}(\ln c_1 * \ln c_2 * \ldots * \ln c_n)}$$

According to the Stolz-Cesaro theorem:

$a_n = \ln c_1 * \ln c_2 * \ldots * \ln c_n$

$b_n = n$

(i) $b_{n+1} > b_n$ $n \epsilon N$ $n+1 > n$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} n = +\infty$

$$L = \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L = \lim_{n\to\infty}\frac{ln\, c_{n+1}}{n+1-n} = \lim_{n\to\infty} lnc_{n+1} = \lim_{n\to\infty}\frac{\ln c_1 + \ln c_2 + \cdots + \ln c_n}{n} = \lim_{n\to\infty} lnc_{n+1}$$

$$e^{\lim_{n\to\infty}\frac{1}{n}(\ln c_1 + \ln c_2 + \cdots + \ln c_n)} = e^{\lim_{n\to\infty} ln\, c_{n+1}} = \lim_{n\to\infty} lnc_{n+1} = \lim_{n\to\infty}(c_{n+1}) = \lim_{n\to\infty}\frac{x_{n+1}}{x_n}$$

The Stolz-Cesaro theorem can be viewed as a generalization of the Cesaro mean but also as a Hopital's rule for sequences.

L'Hopital's Rule. Suppose f and g are differentiable on some interval that has A as an accumulation point, $\lim_{x\to A} g(x) = \infty$ and $\lim_{x\to A}\frac{f'(x)}{g'(x)} = L$. Then:

$$\lim_{x\to A}\frac{f(x)}{g(x)} = L$$

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