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INTEGRAL CALCULATION METHODS

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ABSTRACT

This article discusses methods for calculating integrals. The article discusses in detail the methods for solving integrals of various types. For each method, there are examples of solutions with step-by-step comments. The considered solution algorithm can be applied to any type of integral.

KEYWORDS: *Method of changing of variable, method of integration by parts, geometric meaning of a definite integral.*

INTRODUCTION

While writing this article, the authors was guided by the following outdated article: the article should be understandable and useful for university students, the authors would like it to be useful for teachers as well.

There is a need to master the method of calculating integrals. In this regard, in many universities throughout the country, within the framework of disciplines, separate topics and problems are studied that can be attributed to integral calculations. However, there is still no complete, systematized based on a general methodology. **[1-5]**

The purpose of the article is to improve methods for calculating by solving problems.

Scientific novelty lies in the fact that it contains the problem of improving university education based on methods for calculating integrals. **[6]**

I. Direct integration method

With the help of identity transformations of the integrand, the integral is reduced to an integral, to which the basic integrations are applicable and possible, the table of integrals can be used. **[7]**

II. Differential injection

In the formulas of the indefinite integral, the value dx means that the differential of the variable x is taken. You can use some of the properties of the differential in order to complicate the expression under the differential sign, thereby simplifying the determination of the integral itself. **[8]**

For this the formula is used

$$y'(x)dx = dy(x)$$

If the required function $y(x)$ is absent, sometimes it can be formed by algebraic transformations. **[9]**

III. Integration by change of variable

Let $x = \varphi(t)$, where the function $\varphi(t)$ has a continuous derivative $\varphi'(t)$, and there is a one-to-one correspondence between the variables x and t.

Then the equality

$$\int f(x)dx = \int f\big(\varphi(t)\big)\varphi'(t)dt$$

A certain integral depends on the variable of integration, therefore, if the change of variable is performed, then it is imperative to return to the original integration. **[10]**

Integration by parts

Integration by parts is called integration by the formula

$$\int udv = u \cdot v - \int vdu$$

When finding the function v from its differential dv, one can take any value of the constant of integration C, since it is not included in the final result. Therefore, for convenience, we will take $C = 0$. The use of the formula for integration by parts is advisable in cases, where differentiation simplifies one of the factors, while integration does not complicate the other. **[11,12,13]**

For example. Calculate the integral

$$\int x^7 e^{x^4} dx$$

Solution

Let's take

$x^4 = t$

$4x^3dx = dt$

$dx = \frac{dt}{4x^3}$

$u = t$

$du = dt$

$dv = e^t dt$

$v = e^t$

$\int x^7 e^{x^4} dx = \int x^7 \cdot e^t \cdot \frac{dt}{4x^3} = \frac{1}{4}\int x^4 \cdot e^t \cdot dt = \frac{1}{4}\int t \cdot e^t \cdot dt =$

$= \frac{1}{4} t \cdot e^t - \frac{1}{4}\int e^t dt = \frac{1}{4} te^t - \frac{1}{4} e^t + C = \frac{1}{4} x^4 e^{x^4} - \frac{1}{4} e^{x^4} + C =$

$= \frac{1}{4} e^{x^4}(x^4 - 1) + C$

Now we present methods for solving integrals of different types.

I. Indefinite integral

1. Calculate the integral

$$\int \frac{(x^2+12)dx}{(x\sin x + 4\cos x)^2}$$

Solution

$x\sin x + 4\cos x = \sqrt{x^2+16}\cos(x - \mathrm{arctg}\frac{x}{4})$

Let's make substitutions

$x - \mathrm{arctg}\frac{x}{4} = t$

$\left(1 - \frac{1}{1+\frac{x^2}{16}} \cdot \frac{1}{4}\right) dx = dt$

$dx = \frac{x^2+16}{x^2+12} dt$

$\int \frac{(x^2+12)dx}{(x\sin x + 4\cos x)^2} = \int \frac{(x^2+12)dx}{(x^2+16)\cos^2 t} = \int \frac{x^2+12}{x^2+16} \cdot \frac{x^2+16}{x^2+12} \cdot \frac{dx}{\cos^2 t} =$

$= \int \frac{dx}{\cos^2 t} = \mathrm{tg}\, t + C = \mathrm{tg}(x - \mathrm{arctg}\frac{x}{4}) + C = \frac{\mathrm{tg}\, x - \frac{x}{4}}{1 + \frac{x}{4}\mathrm{tg}\, x} + C =$

$$= \frac{4\sin x - x\cos x}{4\cos x + x\sin x} + C$$

2. Calculate the integral

$$\int \frac{x^2 dx}{(x\sin x+\cos x)^2}$$

Solution

1-way

$x\sin x+\cos x=\sqrt{x^2+1}\cos(x-\operatorname{arctg}\frac{x}{4})$

$x-\operatorname{arctg}x=t$

$dt=\left(1-\frac{1}{1+x^2}\right)dx=\frac{x^2}{1+x^2}dx$

$dx=\frac{1+x^2}{x^2}dt$

$$\int \frac{(x^2+12)dx}{(x\sin x+4\cos x)^2}=\int \frac{x^2}{1+x^2}\cdot\frac{1}{\cos^2 t}\cdot\frac{1+x^2}{x^2}dt=\int\frac{dx}{\cos^2 t}=\operatorname{tg}t+C=$$

$$=\operatorname{tg}(x-\operatorname{arctg}x)+C=\frac{\operatorname{tg}x-x}{1+x\operatorname{tg}x}+C=\frac{\sin x-x\cos x}{\cos x+x\sin x}+C$$

2-way

$u=\frac{x}{\cos x}$

$du=\frac{\cos x+x\sin x}{\cos^2 x}dx$

$v=-\frac{1}{x\sin x+\cos x}$

Notice, that

$(x\sin x+\cos x)'=x\cos x$

Multiplying both the numerator and denominator under the integral function by $\cos x$ we get the following integral **[14]**

$$\int\frac{x}{\cos x}\cdot\frac{x\cos x\cdot dx}{(x\sin x+\cos x)^2}=$$

$$=-\frac{x}{\cos x\,(x\sin x+\cos x)}+\int\frac{\cos x+x\sin x}{\cos^2 x}\cdot\frac{1}{\cos x+x\sin x}dx=$$

$$=-\frac{x}{\cos x\,(x\sin x+\cos x)}+\int\frac{dx}{\cos^2 x}=$$

$$=-\frac{x}{\cos x\,(x\sin x+\cos x)}+\frac{\sin x}{\cos x}+C=$$

$$= \frac{-x + x\sin^2 x + \sin x \cos x}{\cos x\,(x \sin x + \cos x)} + C =$$

$$= \frac{-x + x(1 - \cos^2 x) + \sin x \cos x}{\cos x\,(x \sin x + \cos x)} + C =$$

$$= \frac{-x + x - x\cos^2 x + \sin x \cos x}{\cos x\,(x \sin x + \cos x)} + C = \frac{-x\cos^2 x + \sin x \cos x}{\cos x\,(x \sin x + \cos x)} + C =$$

$$= \frac{\sin x - x \cos x}{x \sin x + \cos x} + C$$

3. Calculate the integral

$$\int (2x^2 + 1)e^{x^2} dx$$

Solution

$u = x$

$du = dx$

$dv = 2xe^{x^2} dx$

$v = e^{x^2}$

$$\int (2x^2 + 1)e^{x^2} dx = \int 2x^2 e^{x^2} dx + \int e^{x^2} dx = \int x \cdot 2xe^{x^2} dx + \int e^{x^2} dx =$$

$$= xe^{x^2} - \int e^{x^2} dx + \int e^{x^2} dx = xe^{x^2} + C$$

4. Calculate the integral

$$\int \frac{dx}{x^6 + 1}$$

Solution

$$\int \frac{dx}{x^6 + 1} = \int \frac{dx}{(x^2)^3 + 1} = \int \frac{dx}{(x^2 + 1)(x^4 - x^2 + 1)} =$$

$$= \int \frac{x^2 + 1 - x^2}{(x^2 + 1)(x^4 - x^2 + 1)} dx = \int \frac{1}{x^4 - x^2 + 1} dx - \int \frac{x^2 dx}{(x^3)^2 + 1} =$$

$$= \frac{1}{2} \int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 - x^2 + 1} dx - \frac{1}{3} \int \frac{d(x^3)}{(x^3)^2 + 1} =$$

$$= \frac{1}{2} \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx - \frac{1}{2} \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx - \frac{1}{3} \operatorname{arctg} x^3 + C, =$$

$$=\frac{1}{2}\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}dx-\frac{1}{2}\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}dx-\frac{1}{3}\operatorname{arctg}x^3+C,=$$

$$=\frac{1}{2}\int\frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+1}-\frac{1}{2}\int\frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-\left(\sqrt{3}\right)^2}-\frac{1}{3}\operatorname{arctg}x^3+C,=$$

$$=\frac{1}{2}\operatorname{arctg}(x-\frac{1}{x})-2\sqrt{3}\ln\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right|-\frac{1}{3}\operatorname{arctg}x^3+C$$

5. Calculate the integral

$$\int\frac{xe^{2x}}{(1+2x)^2}dx$$

Solution

$u=xe^{2x}$

$du=(e^{2x}+2xe^{2x})dx=e^{2x}(1+2x)dx$

$dv=\frac{1}{(1+2x)^2}dx$

$v=-\frac{1}{2(1+2x)}$

$$\int xe^{2x}\cdot\frac{1}{(1+2x)^2}dx=-\frac{xe^{2x}}{2(1+2x)}+\frac{1}{2}\int\frac{e^{2x}(1+2x)}{1+2x}dx=$$

$$=\frac{-xe^{2x}}{2(1+2x)}+\frac{1}{2}\int e^{2x}dx=\frac{-xe^{2x}}{2(1+2x)}+\frac{1}{4}e^{2x}+C=$$

$$=\frac{-2xe^{2x}+e^{2x}(1+2x)}{4(1+2x)}+C=\frac{-2xe^{2x}+e^{2x}+2xe^{2x}}{4(1+2x)}+C=\frac{e^{2x}}{4(1+2x)}+C$$

6. Calculate the integral

$$\int\cos(\ln x)\,dx$$

Solution

$$\int\cos(\ln x)\,dx=\int e^t\cdot\cos t\,dt=e^t\cdot\sin t-\int e^t\cdot\sin t\,dt=$$

$$=e^t\sin t+e^t\cos t-\int e^t\cos t\,dt$$

From here

$$2\int e^t\cdot\cos t\,dt = e^t(\sin t+\cos t)$$

$$\int e^t\cdot\cos t\,dt = \frac{e^t}{2}(\sin t+\cos t)+C$$

$$\int \cos(\ln x)\,dx = \frac{x}{2}(\sin(\ln x)+\cos(\ln x))+C$$

II. Definite integral

7. Calculate the integral

$$\int_0^{\frac{\pi}{2}}\frac{\sin x\,dx}{\sin x+\cos x}$$

Solution

$t=\frac{\pi}{2}-x$

$dt=-dx$

$t=0,\ x=\frac{\pi}{2}$

$t=\frac{\pi}{2},\ x=0$

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{\sin x\,dx}{\sin x+\sin(\frac{\pi}{2}-x)}=-\int_{\frac{\pi}{2}}^{0}\frac{\cos t\cdot dt}{\cos t+\sin t}=$$

$$=\int_0^{\frac{\pi}{2}}\frac{\cos x\,dx}{\sin x+\cos x}$$

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin x\,dx}{\sin x+\cos x}+\int_0^{\frac{\pi}{2}}\frac{\cos x\,dx}{\sin x+\cos x}=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}$$

$$I=\frac{\pi}{4}$$

Means

$$\int_0^{\frac{\pi}{2}}\frac{\sin x\,dx}{\sin x+\cos x}=\frac{\pi}{4}$$

8. Prove that if the function $f(x)$ is even, then

$\int_{-a}^{a}\frac{f(x)}{A^x+1}dx=\int_0^a f(x)dx$ (*)

Solution

Change the variable by the formula $x = -t$, then

$$I = \int_{-a}^{a} \frac{f(x)}{A^x + 1} dx = -\int_{a}^{-a} \frac{f(-t)}{A^{-t} + 1} dt = \int_{-a}^{a} \frac{A^t f(t)}{A^t + 1} dt$$

Therefore

$$2I = \int_{-a}^{a} \frac{f(x)}{A^x + 1} dx + \int_{-a}^{a} \frac{A^x f(x)}{A^x + 1} dx = \int_{-a}^{a} f(x) dx = 2\int_{0}^{a} f(x) dx$$

Whence the equality being proved follows.

9. Calculate the integral

$$\int_{-1}^{1} \frac{x^4}{2^{\sin x} + 1} dx$$

Solution

By formula (*), we have

$$\int_{-1}^{1} \frac{x^4}{2^{\sin x} + 1} dx = \int_{0}^{1} x^4 dx = \left. \frac{x^5}{5} \right|_0^1 = \frac{1}{5}$$

Lemma-1

$$\int_{a}^{b} \frac{f(x)}{f(b + a - x) + f(x)} dx = \frac{b - a}{2}$$

Evidence

$t = b + a - x$

$x = b, t = a$

$x = a, t = b$

$dt = -dx$

$$I = \int_{a}^{b} \frac{f(x)}{f(b + a - x) + f(x)} dx = -\int_{a}^{b} \frac{f(b + a - t)}{f(t) + f(b + a - t)} dt =$$

$$= \int_{a}^{b} \frac{f(b + a - x)}{f(x) + f(b + a - x)} dx$$

$$2I = \int_{a}^{b} \frac{f(x)dx}{f(b + a - x) + f(x)} + \int_{a}^{b} \frac{f(b + a - x)}{f(x) + f(b + a - x)} dx = \int_{a}^{b} dx = b - a$$

$$I = \frac{b - a}{2}$$

10. Calculate the integral

$$\int_2^6 \frac{\sqrt{x}dx}{\sqrt{8-x}+\sqrt{x}}$$

Solution

By Lemma-1, we have

$$\int_2^6 \frac{\sqrt{x}dx}{\sqrt{8-x}+\sqrt{x}} = \int_2^6 \frac{\sqrt{x}}{\sqrt{6+2-x}+\sqrt{x}}dx = \frac{6-2}{2} = 2$$

III. Improper integral

11. Calculate the integral

$$\int_0^\infty \frac{dx}{(1+x^{2021})(1+x^2)}$$

Solution

$$x = \frac{1}{t}$$

$$dx = -\frac{dt}{t^2}$$

$$x = \infty, t = 0$$

$$x = 0, t = \infty$$

$$I = \int_0^\infty \frac{dx}{(1+x^{2021})(1+x^2)} = -\int_\infty^0 \frac{\frac{dt}{t^2}}{\left(1+\frac{1}{t^{2021}}\right)\left(1+\frac{1}{t^2}\right)} =$$

$$= \int_0^\infty \frac{t^{2021}dt}{(1+t^{2021})(1+t^2)} = \int_0^\infty \frac{x^{2021}dx}{(1+x^{2021})(1+x^2)}$$

$$2I = \int_0^\infty \frac{1}{(1+x^{2021})(1+x^2)}dx + \int_0^\infty \frac{x^{2021}dx}{(1+x^{2021})(1+x^2)} =$$

$$= \int_0^\infty \frac{(1+x^{2021})dx}{(1+x^{2021})(1+x^2)} = \int_0^\infty \frac{dx}{1+x^2} = \operatorname{arctg} x|_0^\infty = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

12. Calculate the integral

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\operatorname{tg} x)^{2021}}$$

Solution

Let

$t = \operatorname{tg} x$

$x = \operatorname{arctg} t$

$dx = \frac{dt}{1+t^2}$

$\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\operatorname{tg} x)^{2021}} = \int_0^{\infty} \frac{\frac{dt}{1+t^2}}{1+t^{2021}} = \int_0^{\infty} \frac{dt}{(1+t^{2021})(1+t^2)}$

According to the previous problem, we get that this integral is equal to $\frac{\pi}{4}$.

Lemma-2

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{f(\frac{1}{x})}{x^2} dx$$

Evidence

Let

$x = \frac{1}{t}$

$dx = -\frac{dt}{t^2}$

$x \to \infty,\ t \to 0$

$x \to 0,\ t \to \infty$

Then

$\int_0^{\infty} f(x)dx = -\int_{\infty}^{0} \frac{f\left(\frac{1}{t}\right)}{t^2} dt = \int_0^{\infty} \frac{f(\frac{1}{x})}{x^2} dx$

13. Calculate the integral

$$\int_0^{\infty} \frac{\ln(\frac{x^{13}+1}{x^5+1})}{(1+x^2)\ln x} dx$$

Solution

$I = \int_0^{\infty} \frac{\ln(\frac{x^{13}+1}{x^5+1})}{(1+x^2)\ln x} dx = \int_0^{\infty} \frac{\ln(x^{13}+1) - \ln(x^5+1)}{(1+x^2)\ln x} dx$

Applying Lemma-2, we get

$$\int_0^\infty \frac{\ln(x^{13}+1)-\ln(x^5+1)}{(1+x^2)\ln x}dx = \int_0^\infty \frac{\ln(\frac{1}{x^{13}}+1)-\ln(\frac{1}{x^5}+1)}{x^2(1+\frac{1}{x^2})\ln(\frac{1}{x})}dx =$$

$$= \int_0^\infty \frac{\ln(x^{13}+1)-\ln x^{13}-\ln(x^5+1)+\ln x^5}{(1+x^2)(-\ln x)}dx =$$

$$= -\int_0^\infty \frac{\ln(x^{13}+1)-\ln(x^5+1)-8\ln x}{(1+x^2)\ln x}dx = 8\int_0^\infty \frac{\ln x\,dx}{(1+x^2)\ln x} - I$$

$$2I = 8\int_0^\infty \frac{dx}{1+x^2} = 8\text{arctg}\,x|_0^\infty = 8\cdot\frac{\pi}{2} = 4\pi$$

$$I = 2\pi$$

14. Calculate the integral

$$\int_0^\infty \frac{\sin x}{x}dx$$

Solution

Let

$$I(t) = \int_0^\infty \frac{\sin x}{x}e^{-tx}dx$$

$$I(0) = \int_0^\infty \frac{\sin x}{x}dx$$

$$u = \sin x$$

$$du = \cos x\,dx$$

$$v = -\frac{1}{t}e^{-tx}$$

$$u' = \cos x$$

$$du' = -\sin x\,dx$$

$$v' = -\frac{1}{t}e^{-tx}$$

$$I(t)' = \frac{d}{dt}(\int_0^\infty \frac{\sin x}{x}e^{-tx})dx = \int_0^\infty \frac{\sin x}{x}\cdot(-x)e^{-tx}dx = -\int_0^\infty \sin x\,e^{-tx}dx =$$

$$= \sin x\cdot\frac{e^{-tx}}{t}\Big|_0^\infty - \frac{1}{t}\int_0^\infty \cos x\,e^{-tx}dx =$$

$$= 0 + \frac{1}{t^2}\cos x\,e^{-tx}\Big|_0^\infty - \frac{1}{t^2}\int_0^\infty \sin x\,e^{-tx}dx = -\frac{1}{t^2} - \frac{1}{t^2}I'(t)$$

$$I'(t) = -\frac{1}{t^2} - \frac{1}{t^2} I'(t)$$

$$I'(t) \cdot \left(1 + \frac{1}{t^2}\right) = -\frac{1}{t^2}$$

$$I'(t) \cdot \left(\frac{1 + t^2}{t^2}\right) = -\frac{1}{t^2}$$

$$I'(t) = -\frac{1}{1 + t^2}$$

$$I(t) = -\operatorname{arctg} t + C$$

$$I(t \to \infty) = 0$$

$$I(t \to \infty) = -\operatorname{arctg}(\infty) + C = -\frac{\pi}{2} + C = 0$$

$$C = \frac{\pi}{2}$$

$$I(t) = -\operatorname{arctg} t + \frac{\pi}{2}$$

$$I(0) = \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

15. Calculate the integral

$$\int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx$$

Solution

Let's make the following substitution

$$x = \frac{\pi}{2} - t$$

$$t = \frac{\pi}{2} - x$$

$$dx = -dt$$

$$\int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx = \int_{\frac{\pi}{2}}^{0} \ln\left(\cos\left(\frac{\pi}{2} - t\right)\right)(-dt) = \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = I$$

$$2I = \int_0^{\frac{\pi}{2}} (\ln(\cos x) + \ln(\sin x)) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cdot \cos x)\, dx =$$

$$= \int_0^{\frac{\pi}{2}} \ln(\frac{1}{2} \sin 2x)\, dx = \int_0^{\frac{\pi}{2}} (\ln\frac{1}{2} + \ln(\sin 2x)) dx =$$

$$= -\int_0^{\frac{\pi}{2}} \ln 2\, dx + \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt =$$

$$= -\frac{\pi}{2}\ln 2 + \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt + \int_{\frac{\pi}{2}}^{\pi} \ln(\sin t)\, dt\right) =$$

$$= -\frac{\pi}{2}\ln 2 + \frac{1}{2}I + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin t)\, dt = -\frac{\pi}{2}\ln 2 + \frac{1}{2}I + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\sin(u + \frac{\pi}{2})\right) du$$

$$= -\frac{\pi}{2}\ln 2 + \frac{1}{2}I + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\cos u)\, du = -\frac{\pi}{2}\ln 2 + \frac{1}{2}I + \frac{1}{2}I = -\frac{\pi}{2}\ln 2 + I$$

$$2I = -\frac{\pi}{2}\ln 2 + I$$

$$I = -\frac{\pi}{2}\ln 2$$

16. Calculate the integral

$$\int_0^{\infty} \operatorname{arctg}^2(\frac{1}{x}) dx$$

Solution

By Lemma-2, we obtain the following equality

$$\int_0^{\infty} \operatorname{arctg}^2(\frac{1}{x}) dx = \int_0^{\infty} \frac{\operatorname{arctg}^2(x)}{x^2} dx$$

Integrate the right-hand side of the equality by parts

$$u = \operatorname{arctg}^2 x$$

$$du = 2 \operatorname{arctg} x \cdot \frac{dx}{1 + x^2}$$

$$dv = \frac{1}{x^2} dx$$

$$v = -\frac{1}{x}$$

$$t = \operatorname{arctg} x$$

$$x = \operatorname{tg} t$$

$$dt = \frac{dx}{1 + x^2}$$

$$u = t$$

$$du = dt$$

$dv = \cot t\, dt$

$v = \ln(\sin x)$

$$\int_0^{\infty} \frac{\text{arctg}^2(x)}{x^2} dx = -\frac{\text{arctg}^2(x)}{x}\Bigg|_0^{\infty} + \int_0^{\infty} \frac{2\text{arctg}(x)}{x(x^2+1)} dx = 0 + \int_0^{\infty} \frac{2\text{arctg}(x)}{x(x^2+1)} dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{t}{\text{tg}\, t} dt = 2\int_0^{\frac{\pi}{2}} t \cot t\, dt = 2\left(t\ln(\sin t)|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt \right) =$$

$$= 0 - 2\int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt$$

We know from the previous problem that the integral is

$$\int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt = -\frac{\pi}{2}\ln 2$$

Therefore

$$-2\int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt = -2\cdot\left(-\frac{\pi}{2}\right)\ln 2 = \pi \ln 2$$

17. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4} dx$$

Solution

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4} dx = 2\int_0^{\infty} \frac{\cos(tx)}{x^2+4} dx$$

$$I(t) = 2\int_0^{\infty} \frac{\cos(tx)}{x^2+4} dx$$

$$I(0) = \frac{\pi}{2}$$

$u = tx$

$$x = \frac{u}{t}$$

$$dx = \frac{du}{t}$$

$$I^{'}(t) = -2\int_0^{\infty} \frac{x\sin(tx)}{x^2+4} dx = -2\int_0^{\infty} \frac{x^2\sin(tx)}{x(x^2+4)} dx =$$

$$= -2\int_0^\infty \frac{(x^2+4-4)\sin(tx)}{x(x^2+4)}dx = -2\int_0^\infty \frac{\sin(tx)}{x}dx + 8\int_0^\infty \frac{\sin(tx)}{x(x^2+4)}dx =$$

$$= -2\int_0^\infty \frac{\sin u}{u}du + 8\int_0^\infty \frac{\sin(tx)}{x(x^2+4)}dx = -2\cdot\frac{\pi}{2} + 8\int_0^\infty \frac{\sin(tx)}{x(x^2+4)}dx =$$

$$= -\pi + 8\int_0^\infty \frac{\sin(tx)}{x(x^2+4)}dx$$

Means

$$I^{'}(t) = -\pi + 8\int_0^\infty \frac{\sin(tx)}{x(x^2+4)}dx$$

$$I^{''}(t) = 8\int_0^\infty \frac{x\cos(tx)}{x(x^2+4)}dx = 8\int_0^\infty \frac{\cos(tx)}{x^2+4}dx = 4I(t)$$

We have obtained a homogeneous differential equation of the second order.

$I^{''}(t) = 4I(t)$

$I^{''}(t) - 4I(t) = 0$

Make up a characteristic equation

$k^2 - 4 = 0$

$k_1 = 2,\ k_2 = -2$

General solution equation is equal

$I(t) = c_1e^{2t} + c_2e^{-2t}$

$I(0) = c_1 + c_2 = \frac{\pi}{2}$

$I^{'}(t) = 2c_1e^{2t} - 2c_2e^{-2t}$

$I^{'}(t) = 2c_1 - 2c_2 = -\pi$

$c_1 = 0,\ c_2 = \frac{\pi}{2}$

$I(t) = \frac{\pi}{2}e^{-2t}$

$$I(3) = \int_{-\infty}^\infty \frac{\cos(3x)}{x^2+4}dx = \frac{\pi}{2}e^{-6} = \frac{\pi}{2e^6}$$

18. Calculate the integral

$$\int_0^\infty \frac{1}{1+x^4}dx$$

Solution

1-way

By Lemma-2, we obtain the following equality

$$I=\int_0^\infty \frac{1}{1+x^4}dx=\int_0^\infty \frac{x^2}{1+x^4}dx=\int_0^\infty \frac{x^2dx}{1+2x^2\cos(2\alpha)+x^4}$$

Where $\alpha = \pi/4$

Adding this integral with the last integral we get

$$2I=\int_0^\infty \frac{1+x^2}{1+2x^2\cos(2\alpha)+x^4}dx$$

$$I=\frac{1}{2}\int_0^\infty \frac{1+x^2}{1+2x^2\cos(2\alpha)+x^4}dx$$

Knowing that the integrand function is an even function, that is

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

We get that

$$I=\frac{1}{4}\int_{-\infty}^\infty \frac{1+x^2}{1+2x^2\cos(2\alpha)+x^4}dx$$

Knowing that

$$\cos(2\alpha)=1-2\sin^2(\alpha)$$

$$x^4+2x^2\cos(2\alpha)+1=x^4+2x^2(1-2\sin^2(\alpha))+1=$$

$$=x^4+2x^2+1-4x^2\sin^2(\alpha)=(x^2+1)^2-(2x\sin\alpha)^2=$$

$$=(x^2-2x\sin\alpha+1)(x^2+2x\sin\alpha+1)$$

Since the function $f(x)=-2x\sin\alpha$ is odd, it follows that the function

$$g(x)=-\frac{2x\sin(\alpha)}{(x^2-2x\sin\alpha+1)(x^2+2x\sin\alpha+1)}$$

Then

$$\int_{-\infty}^\infty g(x)dx = \int_{-\infty}^\infty \frac{-2x\sin(\alpha)\,dx}{(x^2-2x\sin\alpha+1)(x^2+2x\sin\alpha+1)}=0$$

$$I=\frac{1}{4}\int_{-\infty}^\infty \frac{(1+x^2)dx}{(x^2-2x\sin\alpha+1)(x^2+2x\sin\alpha+1)}+$$

$$+\frac{1}{4}\int_{-\infty}^\infty \frac{-2x\sin(\alpha)\,dx}{(x^2-2x\sin\alpha+1)(x^2+2x\sin\alpha+1)}=$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \frac{x^2 - 2x\sin(\alpha) + 1}{(x^2 - 2x\sin\alpha + 1)(x^2 + 2x\sin\alpha + 1)}dx \quad = \frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 2x\sin\alpha + 1)}$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{(x + \sin\alpha)^2 + 1 - \sin^2(\alpha)} = \frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{(x + \sin\alpha)^2 + \cos^2(\alpha)} =$$

$$= \frac{1}{4\cos\alpha}\operatorname{arctg}\frac{x + \sin\alpha}{\cos\alpha}\Big|_{-\infty}^{\infty} = \frac{1}{4\cos\alpha}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{4\cos\alpha} = \frac{\pi}{4\cos\frac{\pi}{4}} =$$

$$= \frac{\pi}{4\cdot\frac{\sqrt{2}}{2}} = \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}\pi}{4}$$

2-way

$$I = \int_0^{\infty} \frac{1}{x^4 + 1}dx = \int_0^{\infty} \frac{x^2}{1 + x^4}dx = \int_0^{\infty} \frac{(x^2 + 1 - 1)dx}{1 + x^4} = \int_0^{\infty} \frac{x^2 + 1}{1 + x^4}dx -$$

$$-\int_0^{\infty} \frac{dx}{1 + x^4} = \int_0^{\infty} \frac{x^2 + 1}{1 + x^4}dx - I$$

$$2I = \int_0^{\infty} \frac{x^2 + 1}{1 + x^4}dx$$

Let’s make substitutions

$$t = x - \frac{1}{x}$$

$$dt = \left(1 + \frac{1}{x^2}\right)dx$$

$$x \to \infty,\ t \to \infty$$

$$x \to 0,\ t \to -\infty$$

$$I = \frac{1}{2}\int_0^{\infty} \frac{x^2 + 1}{1 + x^4}dx = \frac{1}{2}\int_0^{\infty} \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2} + x^2}dx = \frac{1}{2}\int_0^{\infty} \frac{d(x - \frac{1}{x})}{(x - \frac{1}{x})^2 + 2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{t^2 + 2} =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{2\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}}\Big|_{-\infty}^{\infty} = \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}\pi}{4}$$

19. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}dx$$

Solution

$$I = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,dx = 2\int_0^{\infty} \frac{\sin^2(x)}{x^2}\,dx$$

$$I(t) = \int_0^{\infty} \frac{\sin^2(tx)}{x^2}\,dx$$

$$I(t) = \frac{\pi}{2}$$

Let's make substitutions

$$u = 2tx$$

$$dx = \frac{du}{2t}$$

$$x = \frac{u}{2t}$$

$$I'(t) = \frac{d}{dt}\left(\int_0^{\infty} \frac{\sin^2(tx)}{x^2}\,dx\right) = \int_0^{\infty} \frac{1}{x^2} 2x \sin(tx)\cos(tx)\,dx = \int_0^{\infty} \frac{\sin(2tx)}{x}\,dx$$

$$= \int_0^{\infty} \frac{\sin u}{\frac{u}{2t}} \cdot \frac{du}{2t} = \int_0^{\infty} \frac{\sin u}{u}\,du = \frac{\pi}{2}$$

(See example 14)

$$I'(t) = \frac{\pi}{2}$$

$$I(t) = \frac{\pi}{2}t + C$$

$$I(1) = \frac{\pi}{2}\cdot 1 + C = \frac{\pi}{2}$$

$$C = 0$$

$$I(t) = \frac{\pi}{2}t$$

$$I(1) = \frac{\pi}{2}$$

$$I = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,dx = 2\int_0^{\infty} \frac{\sin^2(x)}{x^2}\,dx = 2\cdot\frac{\pi}{2} = \pi$$

IV. Definite integral applications

One of the important applications of the definite integral is its use when finding the areas of plane figures. Here are some examples. **[15]**

1.Prove that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}>\ln n,\quad n\in N$$

Solution

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}$$

The sum is the sum of the areas if all $n-1$ quadrangles shown in (Fig.1). Since this sum is the area of the entire stepped figure in (Fig.1), it is strictly greater than

$$\int_1^n \frac{dx}{x}$$

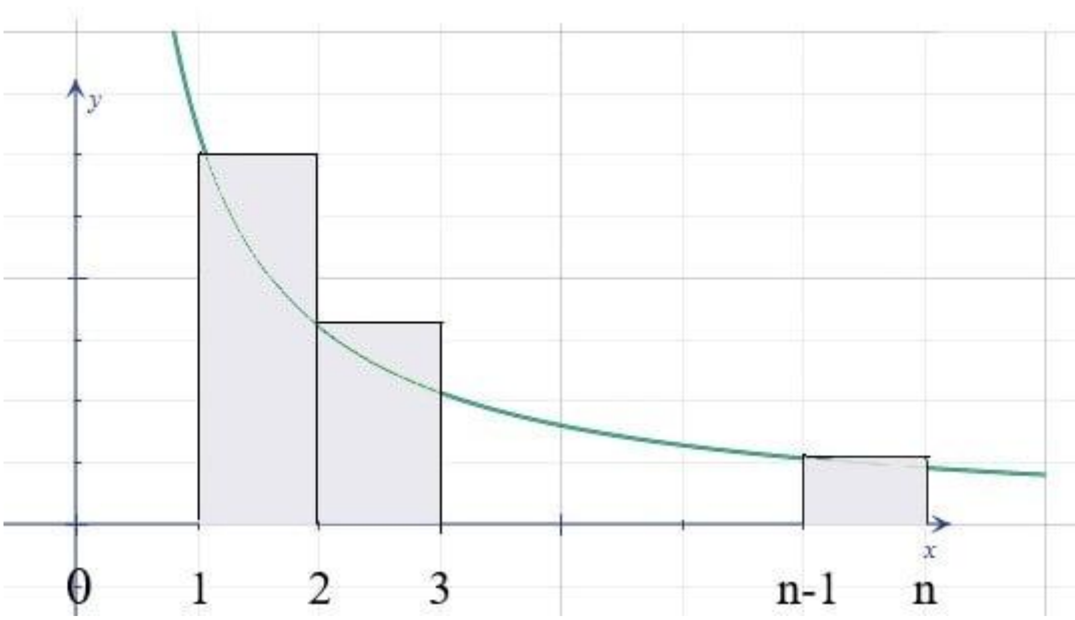

Fig.1

In this way

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}>\int_1^n \frac{dx}{x}=\ln n$$

As required to prove.

2. Prove inequality

$$n!<n^{n+0,5}e^{-n+1},\quad n\in N$$

Solution

Points of the graph of the function $y = \ln x$ with the same abscissas in (Fig.2). Then the area of the curved trapezoid is$A_1B_nA_n$, i.e. $\int_1^n \ln x\, dx$, greater than the sum of the areas of triangle $A_1B_2A_2$ and trapezoids $A_2B_2B_3A_3, \ldots, A_{n-1}B_{n-1}B_nA_n$. So

$$\frac{\ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \cdots + \frac{\ln(n-1) + \ln n}{2} < \int_1^n \ln x\, dx$$

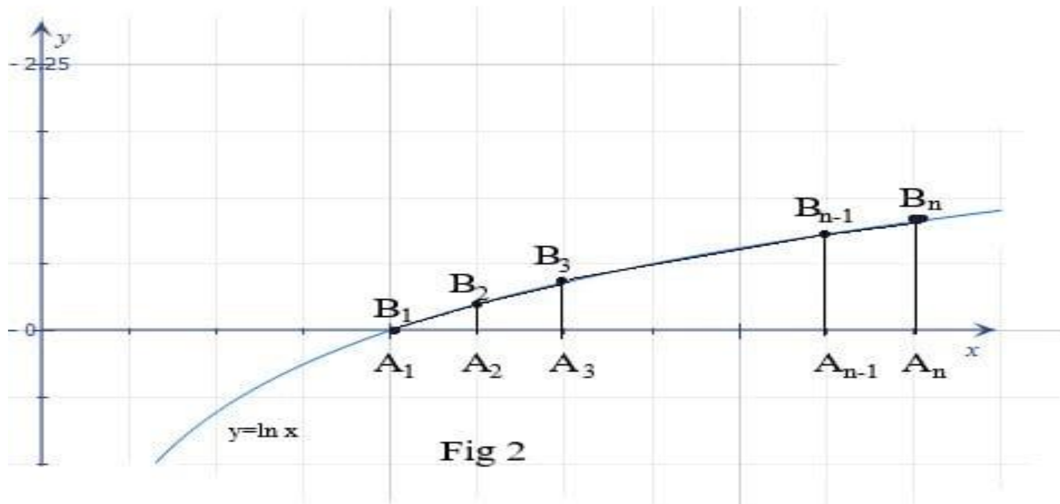

Fig.2

Integrating by parts, we get

$$\int_1^n \ln x\, dx = x \ln x|_1^n - \int_1^n dx = n \ln n - n + 1$$

Hence,

$$\ln 1 + \ln 2 + \cdots + \ln(n-1) + \frac{1}{2}\ln n < n \ln n - n + 1$$

Whence

$\ln n! < (n + 0{,}5) \ln n - n + 1$ or $\ln n! < \ln(n^{n+0{,}5}e^{-n+1})$

And thus inequality

$n! < n^{n+0{,}5}e^{-n+1}$ is proved

3. Prove that

$$\sum_{n=1}^{100} \frac{1}{n\sqrt{n}} \leq 1{,}8 n \in N$$

Solution

Consider the function $f(x) = \frac{1}{x\sqrt{x}}$, $x \geq 1$. Since the function $f(x)$ is decreasing on the interval under consideration, the following inequalities hold

$\frac{1}{2\sqrt{2}} \leq \frac{1}{x\sqrt{x}}$ at $1 \leq x \leq 2$

$\frac{1}{3\sqrt{3}} \leq \frac{1}{x\sqrt{x}}$ at $2 \leq x \leq 3$

…

$\frac{1}{100\sqrt{100}} \leq \frac{1}{x\sqrt{x}}$ at $99 \leq x \leq 100$

Integrating these inequalities, we obtain

$$\frac{1}{2\sqrt{2}} \leq \int_1^2 \frac{1}{x\sqrt{x}}\,dx$$

$$\frac{1}{3\sqrt{3}} \leq \int_2^3 \frac{1}{x\sqrt{x}}\,dx$$

…

$$\frac{1}{100\sqrt{100}} \leq \int_{99}^{100} \frac{1}{x\sqrt{x}}\,dx$$

Adding the resulting inequalities, we find

$$\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots + \frac{1}{100\sqrt{100}} \leq \int_1^2 \frac{dx}{x\sqrt{x}} + \int_2^3 \frac{dx}{x\sqrt{x}} + \cdots + \int_{99}^{100} \frac{dx}{x\sqrt{x}} =$$

$$= \int_1^{100} \frac{1}{x\sqrt{x}}\,dx = -\frac{2}{\sqrt{x}}\Big|_1^{100} = -\frac{2}{10} + 2 = 1{,}8$$

Whence the proved inequality follows.

REFERENCES

1. Antonov NS. Integrative function of training. Modern problems methods of teaching mathematics. Moscow: Education 1985. pp. 25-38
2. Arnold VI. Math and mathematical education in the modern world. Mathematics education. 1977;(2):109-112.
3. Babansky YK. Optimization of learning process. Moscow: 1977.
4. Vasilevky AB. Methods for solving problems. Moscow: "High school"; 1974. 238 p.
5. Genkin GZ. Geometric solutions of algebraic tasks. Mathematics at school. 2001;(7):61-66.
6. Ginguls EG. The development of mathematical abilities of students. Mathematics at school. 1990;(1):14-17.

7. Gordina SV. Methodological basis of the integration of mathematics education the diss. of the candidate ped. science. saransk, 2002. 169p.

8. Dalinger VA. Geometry helps algebra. Mathematics at school. 1996;(4):29-34.

9. Daniluk AY. The theory of the integration of education. Rostov: Publisher Rostov. Ped. 2000. 440 p.

10. Dvoyakovsky PG. About the geometric solution of algebraic tasks. Mathematics at school, 1980;(30):33-35.

11. Osanov AA. The use of the principle of integration in the formation of a holistic view of the world. integration of education

12. Saranzev GV. The methology of teaching methods mathematics. Saransk, 2001. 144 p.

13. Kurlyanchik L, Senderov V. Quantun task. Kvant. 1992;(9):29-31.

14. Demidovich BP. Collection of problems and exercises on mathematical analysis. Moscow 1977, 528 p.

15. Melnikov VV et al. Problems in mathematics" Beginnings of analysis. Moscow; 1990, 608p.