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VOLTAIRE QUADRATIC STOCHASTIC OPERATORS OF A BISEXUAL POPULATION

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ABSTRACT

This article discusses the Voltaire quadratic stochastic operators of the bisexual population and proves it with examples. The concept of a quadratic stochastic operator was first given in the work of S.N. Bernstein [1], devoted to the solution of a mathematical problem related to the theory of heredity. Quadratic operators as an object of research appeared at the turn of the thirties in the works of Ulam [2], where the task was to study the behavior of the trajectories of quadratic operators. The impossibility of creating sufficiently developed analytical methods due to complex and inconvenient iterations in the study of trajectories, and the need for a large number of calculations in the study of exact quadratic operators, did not arouse interest in this problem. The creation of a computer in the forties revived interest in the problem of studying the behavior of the trajectories of quadratic operators. Ulam and his collaborators performed computer calculations for a sufficiently large number of quadratic operators.

KEYWORD: *Quadratic Stochastic Operators, Differential Equations, Mathematical Models Of Genetics, Heritability Coefficient.*

INTRODUCTION

Quadratic stochastic operators appear in very different areas of mathematics and its applications: probability theory, the theory of differential equations, the theory of dynamical systems, mathematical biology, and others.

The theory of quadratic stochastic operators has evolved over 85 years and many papers have been published.

The quadratic stochastic operator (QSO) by population freedom has the following meaning:

The Main Findings and Results

Consider a certain biological population, i.e. a community of organisms closed with respect to reproduction. Suppose that each individual in the population belongs to some unique of n varieties 1, 2, 3,..., n. The scale of varieties (traits, phenotypes, genotypes) should be such that the varieties of parents *i* and *j* uniquely determine the probability of each variety *k* for the immediate descendant of the first generation. Let us designate this probability ("Heritability Coefficient") by $P_{ij,k}$. Obviously in this case the conditions:

$$P_{ij,k} \geq 0, \sum_{k=1}^{n} P_{ij,k} = 1, \quad \text{for all } i,j,k$$

Suppose the population is so large that the frequency fluctuations can be neglected. Then it states can be described by a set of $x = (x_1, x_2, x_3 \dots, x_n)$ probabilities of varieties. Those, x_i is the fraction of species *i* in the population.

With the so-called panmixia, or accidental crossing in a fixed state $x = (x_1, x_2, x_3 \dots, x_n)$ parent bets *i* and *j* are formed with probability $x_i x_j$ and, therefore,

$$x_k^{'} = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j \qquad (1)$$

will be fully likely to among immediate descendants.

Lots of $S^{n-1} = \{x = (x_1, x_2, \dots, x_n) : \ x_i \geq 0, \quad \text{i=1,2,...,n}, \quad \sum_{i=1}^{n} x_i = 1\}$ (2)

called $n-1$ - dimensional simplex and, since $\sum_{k=1}^{n} x_k^{'} = 1$ и $x_k^{'} \geq 0$, then the mapping (2) is called a quadratic stochastic operator, takes the simplex S^{n-1} into itself.

where $P_{ij,k}$ is the inheritance coefficient satisfy the conditions:

$$P_{ij,k} \geq 0, \qquad \sum_{k=1}^{n} P_{ij,k} = 1 \quad , \quad \text{i,j,k}. \qquad (3)$$

Among the mathematical models of genetics, models generated by quadratic operators play an important role.

The trajectory $\{(x^{(t)})\}_{t=0}^{\infty}$, $t = 1,2,\dots$ for $x^{(0)} \in S^{n-1}$ under the action of QSO (2) is determined as follows

$$x^{(n+1)} = V\left(x^{(n)}\right), n = 0,1,2,\dots$$

One of the main problems for a given operator in mathematical biology is to study the asymptotic behavior of trajectories. This problem was completely solved for the Voltaire CSR, which are determined by equalities (1), (3) and the additional assumption

$$P_{ij,k} = 0, \;\; если\ k \,\overline{\in}\, \{i,j\} \quad (4)$$

In this paper, we consider the quadratic stochastic operators of the bisexual population

Definitions 1: Let $F = \{F_1, F_2, F_3 \dots, F_n\}$ be a set of feminine type, $M = \{M_1, M_2, M_3 \dots, M_v\}$ is a set of masculine type. The state of a population is a pair of probability distributions

$x = \{x_1, x_2, x_3 \dots, x_n\}$ – and $y = \{y_1, y_2, y_3 \dots, y_v\}$ – on the sets F and M.

$$x_i \geq 0 \qquad \sum_{i=1}^{n} x_i = 1 \quad (5)$$

$$y_i \geq 0 \quad \sum_{i=1}^{v} y_i = 1$$

The state space of a given population is $S^{n-1} \times S^{v-1}$Cartesian product of (n-1) dimensional simplex S^{n-1} by (v-1) dimensional simplex S^{v-1}.

Population differentiation is called hereditary if, for any state *(x, y)* in generation *G*, the state *(x ', y')* is uniquely determined, arising in the next generation *G '*by crossing and selection.

Mapping $W: S^{n-1} \times S^{v-1} \to S^{n-1} \times S^{v-1}$, mapping *(n-1) * (v-1)* dimensional cartesian product defined by equality

$$(x', y') = W(x, y), \;\; (x, y) \in S^{n-1} \times S^{v-1} \quad (6)$$

Called evolutionary operator: In coordinates, it turns into a system of equalities

$$x_i' = f_i(x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_{v,}),$$

$$1 \leq i \leq n,$$

$$(7)$$

$$y_k' = g_k(x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_{v,}),$$

$$1 \leq k \leq v,$$

Which are also called Evolutionary. Display (7) for any initial state (x^0, y^0) uniquely determines the trajectory

$$\{(x^{(t)}, y^{(t)})\}_{t=0}^{\infty} : (x^{(t+1)}, y^{(t+1)}) = \qquad (8)$$

$$W((x^{(t)}, y^{(t)})) = W^{(t+1)}((x^{(0)}, y^{(0)})), \qquad t = 1,2, \dots$$

The set of limit points of a trajectory starting at point (x^0, y^0) is called its limit set and is denoted by $\omega(x^0, y^0)$.

We derive the evolutionary equations of a bisexual population. The initial data for this are the heredity rates $P_{ik,j}^{(f)}, P_{ik,j}^{(m)}$.

The value $P_{ik,j}^{(f)}$ is defined as the probability of the birth of a female offspring of type $F_j,\ 1 \le j \le n.$ in a mother of type $F_j,\ 1 \le j \le n$, and a father of type $M_k,\ 1 \le k \le v$ Similarly, $P_{ik,j}^{(m)}, 1 \le i \le n, 1 \le k \le v$ is clearly defined,

$$P_{ik,j}^{(f)} \ge 0, \ \sum_{j=1}^{n} P_{ik,j}^{(f)} = 1$$

$$P_{ik,j}^{(m)} \ge 0, \ \sum_{j=1}^{n} P_{ik,j}^{(m)} = 1 \qquad (9)$$

Heredity coefficients take into account, for example, factors such as the recombination process, gamete selection, mutations and differential fertility.

Let (x, y) be a state in generation *G. (x ', y')* - arising in the next generation G 'at the moment of its content, the probabilities of types are found according to the formula of total probabilities:

$$W: \begin{cases} x_j' = \sum\limits_{i,k=1}^{n,v} P_{ik,j}^{(f)} x_i y_k, 1 \le j \le n, \\ y_l' = \sum\limits_{i,k=1}^{n,v} P_{ik,l}^{(f)} x_i y_k, 1 \le l \le v, \end{cases} \qquad (10)$$

Definition 2: An evolutionary operator (10) is called the Voltaire quadratic stochastic operators of a bisexual population (VQSOBP) if the heredity coefficients (9) satisfy the condition

$$P_{ik,j}^{(f)} = 0,\ \textit{если}\ \ j\bar{\epsilon}\{i,k\}, 1 \le i,j \le n,\ 1 \le k \le v, \tag{11}$$

$$P_{ik,l}^{(m)} = 0,\quad \textit{если}\ l\bar{\epsilon}\{i,k\},\quad 1 \le i \le n,\quad 1 \le k,l \le v,$$

To be specific, let's assume $n < v$. Then it is easy to see that an arbitrary VKSODP has the form

$$W: \begin{cases} x_j' = x_j\left(1 + \sum_{k \ne j, k=1}^{v} \left(P_{jk,j}^{(f)} - 1\right) y_k\right) + y_j\left(\sum_{k \ne j, k=1}^{n} P_{kj,j}^{(f)} x_k\right), 1 \le j \le n, \\ y_l' = y_l\left(1 + \sum_{k \ne l, k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) x_k\right) + x_l\left(\sum_{k \ne l, k=1}^{v} P_{lk,l}^{(m)} y_k\right), 1 \le l \le n, \\ y_l' = y_l\left(1 + \sum_{k \ne l, k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) x_k\right), \qquad n \le l \le v, \end{cases} \tag{12}$$

Many fixed points. The set of Fix (W) -fixed points of the VQSOBP, the fixed points of the operator W are solutions of the equation

$W(x, y) = (x, y)$ those.

$$x_j = x_j\left(1 + \sum_{k \ne j, k=1}^{v} \left(P_{jk,j}^{(f)} - 1\right) y_k\right) + y_j\left(\sum_{k \ne j, k=1}^{n} P_{kj,j}^{(f)} x_k\right), 1 \le j \le n,$$

$$y_l = y_l\left(1 + \textstyle\sum_{k \ne l, k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) x_k\right) + x_l\left(\textstyle\sum_{k \ne l, k=1}^{v} P_{lk,l}^{(m)} y_k\right), 1 \le l \le n, \tag{13}$$

$$y_l = y_l\left(1 + \sum_{k \ne l, k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) x_k\right), \qquad n \le l \le v,$$

In the system of linear equations (13) in the right n-1 equations, we replace x $_n$ with

$$x_n = 1 - \sum_{i=1}^{n-1} x_i$$

and get

$$\left[\sum_{k \ne j, k=1}^{v} \left(1 - P_{jk,j}^{(f)}\right) y_k + y_j P_{nk,j}^{(f)}\right] x_j + \tag{14}$$

$$+\sum_{k\neq j,k=1}^{n-1}\left(P_{nj,j}^{(f)}-P_{kj,j}^{(f)}\right)y_j x_k = y_j P_{nj,j}^{(f)}, \qquad 1\leq j\leq n-1,$$

From a system of linear equations (with respect to $x_1, x_2, x_3, \ldots, x_{n-1}$) (14) using Cramer's method, we find the unknowns $x_1, x_2, x_3, \ldots, x_{n-1}$.

Let us denote by $C=(c_{ij})_{i,j=1}^{n-1}$ the matrix consisting of the coefficients of system (14) and by $C^s=(c_{ij}^s)_{i,j=1}^{n-1}$ the matrix obtained from the matrix $\boldsymbol{C}$ by replacing the s-th column with the column of free terms, where

$$c_{ij}=\begin{cases}\sum_{k\neq j,k=1}^{v}\left(1-P_{jk,j}^{(f)}\right)y_k+y_jP_{nj,j}^{(f)}, если \quad i=j\\ \left(P_{nj,j}^{(f)}-P_{ij,j}^{(f)}\right)y_j, \qquad если \ i\neq j\end{cases} \tag{15}$$

$$c_{ij}^s=\begin{cases} c_{ij}, если \quad i\neq s\\ y_jP_{nj,j}^{(f)}, если\ i=s, 1\leq i,j\leq n-1\end{cases} \tag{16}$$

Determinants are denoted by $\Delta=\det(C), \Delta_s=\det(C^s)$

From (15) and (16) it follows that each element of the s-row of the determinant Δ_s – contains the factor y_s . Then the determinant Δ_s – can be written as $\Delta_s = y_s\overline{\Delta_s}$ Where $\overline{\Delta_s}=det(\overline{C^{(s)}})$, $\overline{C^{(s)}}=\left(\overline{c_{ij}^s}\right)$ и

$$\overline{c_{ij}^s}=\begin{cases} c_{ij}, \qquad если \quad i\neq s\\ P_{nj,j}^{(f)}, если\ i=s, 1\leq i, \quad j\leq n-1\end{cases}$$

Therefore, if $\Delta\neq 0$, the solution to system (14) is unique and has the form

$$x_s=\frac{\Delta_s}{\Delta}=y_s\frac{\overline{\Delta_s}}{\Delta}, \quad 1\leq s\leq n-1 \quad (17)$$

Using (17), from (13) we obtain

$$y_l=y_l\left(1+\sum_{k\neq l,k=1}^{n}\left(P_{kl,l}^{(m)}-1\right)y_k\frac{\overline{\Delta_k}}{\Delta}+\frac{\overline{\Delta_l}}{\Delta}\left(\sum_{k\neq l,k=1}^{v}P_{lk,l}^{(m)}y_k\right)\right)(18)$$

We denote

$$A_l(y_1y_2,\dots,y_v)=\sum_{k\neq l,k=1}^{n}\left(P_{kl,l}^{(m)}-1\right)y_k\frac{\overline{\Delta_k}}{\Delta}+ \qquad (19)$$

$$+\frac{\overline{\Delta_l}}{\Delta}\left(\sum_{k\neq l,k=1}^{v}P_{lk,l}^{(m)}y_k\right),1\leq l\leq v$$

Then (18) takes the form

$$y_l=y_l(1+A_l(y_1y_2,\dots,y_v)=y_l(1+A_l(y),1\leq l\leq v \quad (20)$$

Thus, the problem of describing the fixed points of the operator W is reduced to finding the fixed points of the operator V: $S^{v-1}\to S^{v-1}$ of a certain right-hand side of (20), i.e. $V\colon y_l^{'}=y_l(1+A_l(y)),\quad 1\leq l\leq v \quad (21)$

Comment: The vertices of the simplex S^{v-1} will be fixed points of the operator V. solutions of the system of equations (20).

Consider the mapping A=$(A_1A_2,\dots,A_v)\colon S^{v-1}\to R^v$, Where $A_i,i=\overline{\iota,\dots,v}$ determined by the formula (19). Let be $I=\{1,\dots,v\}$ and $\alpha\subset I$ Arbitrary subset. Lots of

$$L_\alpha=\{y\in S^{v-1}\colon y_k=0,k\,\overline{\in}\,\alpha\}$$

Called the faces of the simplex, Lots of

$$int(L_\alpha)=\{y\in L_\alpha\colon y_k=0,k\in\alpha\}$$

is called the relative interior of the face L_α.

For vectors $x,y\in R^v x\Box_\alpha y$, we put if $x_i>y_i$ at $i\epsilon\alpha$ and $x_i\geq y_i$ at $i\,\overline{\in}\,\alpha$. If $\alpha=\emptyset$, then write $x\Box y$.

Theorem.1. $A=(A_1,\dots,A_v)\colon S^{v-1}\to R^v$ continuously

2. $A_l(y)\geq -1,\mathrm{I}=(-1,-1,\dots,-1)$ for anyone $y\in S^{v-1}$

3. $(A(y),y)=0$ for each $y\in S^{v-1}$

4. For any $\alpha\subset I$, performed $A(y)\Box_\alpha -I$ for any $y\in int(L_\alpha)$

Proof: 1. It follows from the fact that $\Delta\neq 0$.

2. For anyone *l*. $1\leq l\leq v$ and anyone $y\in S^{v-1}$ taking into account (11) and (17), we have

$$A_l(y) = \sum_{k\neq l,k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) y_k \frac{\overline{\Delta_k}}{\Delta} + \frac{\overline{\Delta_l}}{\Delta}\left(\sum_{k\neq l,k=1}^{v} P_{lk,l}^{(m)} y_k\right) \geq$$

$$\geq \sum_{k\neq l,k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) x_k \geq - \sum_{k\neq l,k=1}^{n} x_k \geq -1, \quad 1 \leq l \leq v$$

3. Using (17) and (21), we obtain

$$(A(y),y) = \sum_{l=1}^{v}\left[\sum_{k\neq l,k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) y_k \frac{\overline{\Delta_k}}{\Delta} + \frac{\overline{\Delta_l}}{\Delta}\left(\sum_{k\neq l,k=1}^{v} P_{lk,l}^{(m)} y_k\right)\right] y_{l=}$$

$$= \sum_{l=1}^{v}\left[\sum_{k\neq l,k=1}^{n} \left(P_{kl,l}^{(m)} - 1\right) x_k y_l + x_l\left(\sum_{k\neq l,k=1}^{v} P_{lk,l}^{(m)} y_k\right)\right]$$

$$= \sum_{l=1}^{v}(y_l^{'} - y_l) = \sum_{l=1}^{v} y_l^{'} - \sum_{l=1}^{v} y_l = 0$$

4. Since $\Delta \neq 0$ and $y \in int(L_\alpha)$ are from property 2. It follows that $A_l(y) > -1$ for every $y \in int(L_\alpha)$.

The following corollaries follow from the theorem.

Consequence 1*:* $|Fix(W)| = |Fix(V)|$, Where $|.|$ denotes the number of elements in a set.

Consequence 2*:* The operator V(cm(21)) is a Voltaire-type operator

Definition 3: A fixed point $x \in Fix(W)$ is called an isolated fixed point of the operator (12) if there exists a neighborhood of the point *x* in which there are no fixed points other than *x*.

CONCLUSION

In short, since W is a continuous compact operator, it has at least one fixed point. Therefore, if $\Delta = 0$ then system (14) has an infinite set of solutions, and some of them are not isolated,

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